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Predicting Fully-developed Channel Flow with Zero-equation Model

M. M. Rahman[1*], K. Hasan[1], Wenchang Liu[1], and Xinming Li[1]

[1,2,3]Hangzhou Dianzi University, School of Mechanical Engineering, 310018 Hangzhou, China

emails: *[*1]mizanur_rahman@hdu.edu.cn; [2]m.khasan1100@gmail.com; [3]chang910521@foxmail.com;*

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ABSTRACT

A new zero-equation model (ZEM) is devised with an eddy-viscosity formulation using a stress length variable which the structural ensemble dynamics (SED) theory predicts. The ZEM is distinguished by obvious physical parameters, quantifying the underlying flow domain with a universal multi-layer structure. The SED theory is also utilized to formulate an anisotropic Bradshaw stress-intensity factor, parameterized with an eddy-to-laminar viscosity ratio. Bradshaw's structure-function is employed to evaluate the kinetic energy of turbulence k and turbulent dissipation rate $\boldsymbol{\varepsilon}$. The proposed ZEM is intrinsically plausible, having a significant impact on the prediction of wall-bounded turbulence.

NOMENCLATURE

C_μ	eddy-viscosity coefficient
DNS	Direct Numerical Simulation
k	turbulent kinetic energy
l_{12}	stress length
R_T	eddy-to-laminar viscosity ratio
S	mean strain-rate
SED	Structural ensemble dynamics
SST	shear stress transport
u_i	Cartesian velocity components
u_T	friction velocity
Y	wall distance
y^+	$u_\tau y/\nu$
ZEM	zero-equation model
γ	compensated stress length
p	pressure
R_b	stress-intensity factor
Re	Reynolds number
ε	turbulent dissipation rate
θ	momentum thickness
κ	von-Karman constant
μ, μ_T	laminar & turbulent eddy viscosities
ν	laminar kinematic viscosity
ρ	density
ω	specific dissipation rate Subscript
i, j	variable quantities
T	turbulent condition

1. INTRODUCTION

Due to the lack of a relevant existing theory which concentrates on the physical understanding of wall turbulence, the framework of the RANS (Reynolds-averaged Navier-Stokes) turbulence model takes a pivotal role (Durbin, 2018) on this deficiency. The formulation of turbulence model typically becomes complicated when introducing correlation artefacts to predict a new flow with original features; correlation terms having dimensional argument retain several functions or coefficients without physical interpretations. In principle, the concept of wall turbulence with its universal structure can avoid a large amount of empiricism involved in wall-bounded flows. Unfortunately, the Prandtl mixing length theory (Prandtl, 1925), von Karman log-law argument (Segalini *et al.*, 2013) and Townsend similarity hypothesis (Townsend, 1976) are not fully capable of providing a consistent route to the development of a zero-equation model (ZEM). The current research convokes such an alternative to developing a plausible algebraic model, resorting to a well-established wall-turbulence phenomenon.

Using a symmetry-based approach, the SED (structural ensemble dynamics) theory (She *et al.*, 2009, 2010, 2017) has been recently proposed. The concept applies an invariant wall restriction on turbulence eddies in the framework of a generalized Lie-group dilation invariance (LDI) to the stress length l_{12}, estimating scales of eddies. The SED theory speculates a multi-layer formulation for l_{12} with a fully-developed channel flow. The analytic solution preserves the information of the entire flow domain with a four-layer structure; four layers consist of a viscous sublayer, buffer layer, bulk flow region (containing log-layer) and core layer. The four-layer structure precisely characterizes the total flow field and captures the genuine similarity image of wall turbulence with an increase in Reynolds number. The SED theory is capable of predicting its universal analytic etiquette sticking to an LDI principle, which is apparently universal in the presence of a wall. Remarkably, the variations in multi-layer parameters represent disparities among various layers of a physical flow domain.

A zero-equation (an algebraic) turbulence model is developed in the current study, adhering to the impressive multi-layer physics of wall turbulence. The turbulent kinematic eddy viscosity is calculated as $\nu_T = l_{12}^2 S$, where S is the strain-rate invariant. Bradshaw's stress-intensity factor [8] is formulated using the SED theory which is parameterized with an eddy-to-laminar viscosity ratio; the resulting structure-function can reasonably predict the kinetic energy of turbulence k and turbulent dissipation rate ε. Predictions from the widely-used SST (shear-stress-transport) *k-ω* turbulence model [9] are compared with those of the ZEM. Results demonstrate that the ZEM performs better than the SST model. A remarkable achievement can be ascribed to the current work in a way that the wall turbulence with a refined physical multi-layer description has prevailed over the drawbacks of previously developed algebraic turbulence models (Prandtl, 1925; Segalini *et al.*, 2013; Townsend, 1976; Wilcox, 2006).

2. GOVERNING EQUATIONS

The stress length of SED theory has specified the turbulent eddy-viscosity in the new algebraic model. The current model begins from RANS equations which deal with the mean conservation of mass and momentum. RANS equations can be represented as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho u_i\right) = 0 \tag{1}$$

$$\frac{\rho u_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho u_i u_j\right) = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} \tag{2}$$

where u_j is the *jth* component of velocity, x_j represents Cartesian coordinates, ρ implies the fluid density and p denotes the pressure. The Boussinesq approximation can be applied to relate total stresses σ_{ij} with the mean strain-rate tensor S_{ij} as:

$$\sigma_{ij} = 2\left(\mu + \mu_T\right)\left(S_{ij} - \frac{1}{3} S_{kk}\delta_{ij}\right), S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3}$$

where the Kronecker's delta functionis indicated by δ_{ij} ; δ_{ij} = 1 for $i = j$ and δ_{ij} = 0 for $i \neq j$. Laminar and turbulent viscosities are designated by μ and μ_T , respectively. According to the SED theory, μ_T can be defined using the stress length function l_{12} as:

$$\mu_T = \rho l_{12}^2 S$$

where the mean strain-rate invariant S is given by S = $\sqrt{2S_{ij}S_{ij}}$. A sub-concept of the SED theory, regarded as the order-function formula of She *et al.* (2010) describes complex systems with multi-layer structures; the wall turbulence belongs to this group. According to the SED theory, a well-defined set of functions in the framework of multi-products (i.e., order functions) is used to format the multi-layer structures. The order function is deduced from a quantitative analysis of the generalized LDI, established with the presence of wall. The order-function generic form can be given as:

$$\phi = c_0\left(\frac{y}{a_0}\right)^{c_0/b_0} \prod_{i=1}^{n}\left[1+\left(\frac{y}{a_i}\right)^{b_i}\right]^{c_i/b_i} \tag{5}$$

where a, b and c can be adjustable constants; φ is an order function having complex multi-layer structures, parameterized with a variable y and n is the number of products. In fact, the spatial variation of φ involves multiple transitions, from one layer to another. Some soft-wares like MATLAB or Mathematica can assist the curve-fitting process. The SED hypothesis speculates that in the fully-developed turbulent boundary layer, a multi-layer form is associated with the stress length l_{12}. It provides (She, 2017):

$$l_{12}^{+} = l_0\left(\frac{y^+}{9.7}\right)^{3/2}\left[1+\left(\frac{y^+}{9.7}\right)^4\right]^{1/8}\left[1+\left(\frac{y^+}{41}\right)^4\right]^{-1/4} \tag{6}$$

$$\frac{1-r^4}{4\left(1-r\right)}\left[1+\left(\frac{0.27}{r}\right)^2\right]^{1/4}$$

where $y^{+} = yu_{\tau}/\nu$ is the dimensionless wall-distance with the friction wall-velocity $u_{\tau} = \sqrt{(\nu S)_{w}}$ which is well-defined for attached and mild separated flows with $S_{w} > 0$; the kinematic laminar viscosity is given by $\nu = \mu/\rho$. In addition, $r = 1 - y/\delta = 1 - y^{+}/\mathrm{Re}_{\tau}$, where δ is the channel half-width and $\mathrm{Re}_{\tau} = \delta u_{\tau}/\nu$ signifies friction Reynolds number. Equation (6) retains a canonical four-layer structure of wall turbulence, consisting of a viscous sublayer, buffer layer, bulk zone and core layer for a fully-developed channel flow. They remain apart from each other by empirically evaluated layer thicknesses, respectively $y^{+}_{sub} = 9.7, y^{+}_{buf} = 41$ and $r_{core} = 0.27$. In other words, the viscous sublayer ending at $y^{+}_{sub} = 9.7$ is the first layer adjacent to the wall; afterward, the buffer layer ends at $y^{+}_{buf} = 41$.

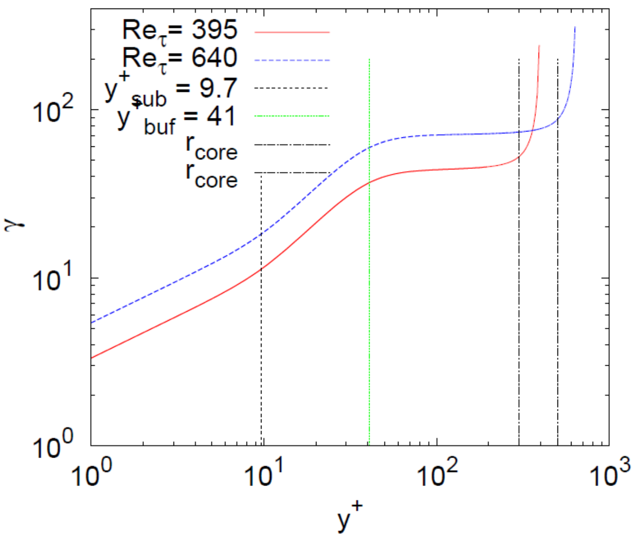

Figure 1: Compensated plot of $\gamma = l^{+}_{12}/(1 - r^{4})$. The bulk layer (between y^{+}_{buf} and r_{core}) retains almost constant values. Dashed lines show thicknesses of viscous sublayer $(y^{+}_{sub} = 9.7)$, buffer layer $(y^{+}_{buf} = 41)$ and core layer $(r_{core} = 0.27)$

The core layer expands from the centre-line to a core layer thickness of $r_{core} = 0.27$ and the bulk-flow region consists of the remaining flow domain. In principle, the need for defining overlap region is avoided by the structure of geometry-dependent bulk-flow, i.e., $(1 - r^4)$. It is not difficult to verify that for $y^+ > 41$, the celebrated linear law $l^{+}_{12} \approx ky^{+}$ is obtained as a matching function between inner and outer regions with $l_{0} \approx 9.7^{2}\kappa/41 \approx 1.0$, where the Karman constant $\kappa = 0.45$ has been chosen from the original SED theory.

Near the centre $r \le r_{core}$ (with r = 1 − y/δ is the distance from the channel centre-line), a core layer has been defined wherein dissipation and pressure-strain are balanced by turbulent transport that takes over production (She, 2017). In principle, such shifts in the balancing mechanism induce the four-layer structure as mentioned earlier. The multi-layer structure can also be observed in the compensated plot (divided by $1-r^4$), as displayed in Figure 1; core layers are at y^{+}_{core} = (300; 500) for Re_{τ} = (395; 640). Direct Numerical Simulation (DNS) data are available from (Mansour *et al.*, 1988; Kawamura *et al.*, 1999).

The current study deals with the zero-equation model (ZEM). However, turbulence quantities can be evaluated from turbulence fluctuations using unsteady RANS computations.

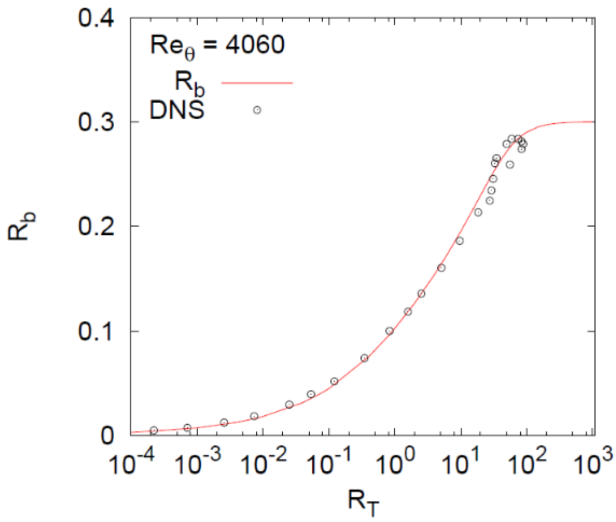

Figure 2: Stress-intensity parameter for flat-plate boundary layer flow at Re_{θ} = 4060

An extension of the Bradshaw hypothesis (Bradshaw *et al.*, 1967) from free shear flows to wall-bounded flows convokes a plausible parameter, resolving near-wall anisotropy of turbulence. The Bradshaw parameter $R_{b} = |-uv|/k \approx \sqrt{C_{\mu}}$, where $|-uv|$ is the principal shear stress, k implies the kinetic energy of turbulence and C_{μ} = 0.09. It is also believed that the SED theory is precisely accurate to formulate a variable Bradshaw structure-function R_{b} in the vicinity of the boundary layer. In Reference [12], R_{b} is fabricated as a function of $R_{T} = \sqrt{k}\,y/\nu$. Nevertheless, the present formulation necessitates R_{b} as a function of $R_{T} = \nu_{T}/\nu$ (eddy-to-laminar viscosity ratio, where $\nu_{T} = \mu_{T}/\rho$ is the kinematic eddy-viscosity) to empirically evaluate k and dissipation-rate ε . Applying the curve-fitting approximation (Equation (5)), the structure parameter R_{b} is obtained after the calibration with DNS data for a fully-developed turbulent flat-plate boundary-layer flow (Schlatter & Orlu, 2010):

$$R_{b} = \frac{C_{1}R_{T}^{0.4}}{(1.0 + R_{T})^{0.16}(1.0 + C_{2}^{2}R_{T}^{2})^{0.12}} \tag{7}$$

where $C_1 = C_\mu^{0.9}$, $C_2 = C_\mu / 5.0$. As $R_T \to \infty$, $R_b \approx C_1 / C_2^{0.24} \approx \sqrt{C_\mu}$. The present work has three products with n = 2. As can be observed from Figure 2, Eq. (7) has a good correspondence with DNS data (Schlatter & Orlu, 2010) of a fully-developed turbulent flat-plate boundary layer at a slice of Re_θ = 4060 (with wake-layer DNS data mostly excluded), where Re_θ signifies the momentum-thickness Reynolds number. The dimensionless turbulent viscosity is calculated as $\nu_T^+ = \left|-uv^+\right| / \left(du^+/dy^+\right)$ using the DNS data. The structure parameter $R_b = \nu_T \tilde{S} / k$ can be employed to calculate k and ε as:

$$k = \frac{\nu_T \tilde{S}}{R_b + C}, \quad \varepsilon = R_b k \tilde{S} \tag{8}$$

where C = 0.01 is used to avoid the near-wall singularity and ε vanishes at the solid wall. It is worth mentioning that on a channel-flow centre-line or outside a boundary layer, the strain-rate invariant S may approach zero. Therefore, the free-stream strain-rate correction S_α with a non-vanishing identity can be convoked as a remedy; it can be approximated from a nearly homogeneous shear flow of Champagne *et al.* (1970) as presented in Equation (9) where the added S_α has a tiny numerical effect.

$$\tilde{S} = \sqrt{S^2 + S_\alpha^2}, \quad S_\alpha = \frac{1}{C_\mu} s^{-1} \tag{9}$$

3. COMPUTATIONS OF FULLY-DEVELOPED TURBULENT CHANNEL FLOW

Fully-developed turbulent channel flows at Re_τ = (395; 640) are simulated to assess the model competency in reproducing near-wall turbulence. Computations are carried out in the half-width of a channel, employing a 1-D (one-dimensional) RANS solver. A non-uniform 1 × 64 grid resolution for Re_τ = 395 and 1×128 grid resolution for Re_τ = 640 are presumably adequate to accurately predict characteristics of the flow (e.g., producing almost grid-independent solutions). To assure the viscous sublayer resolution, the first near-wall cell height is located at $y^+ \approx$ 0.3 to assure the viscous sublayer resolution. A cell-centred finite-volume approach is utilized along with SIMPLE algorithm. Predictions of the present zero-equation model (ZEM) are compared with the well-known Menter SST (shear-stress-transport) model (Menter, 1994).

For a 1-D incompressible flow, the streamwise (x-direction) mean momentum equation can be given as:

$$\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left[\left(\nu + \nu_T\right)\frac{\partial u}{\partial y}\right] = 0 \tag{10}$$

where the positions of lower and upper walls of the channel are indicated by $y = (-h;\ h)$. Since the mean flow field has a 1-D identity, the axial gradient of pressure $\partial p / \partial x$ remains constant, and the continuity constraint $\partial u_i / \partial x_i$ = 0 is naturally satisfied. However, $\partial p / \partial x$ must be computed as a part of the solution method since the pressure gradient is not known a priori. The pressure-velocity correction method (Rahman *et al.*, 1996, 1997) is an appropriate choice to solve the problem. The scheme continually updates the velocity and axial pressure gradient as long as the mass imbalance is minimized.

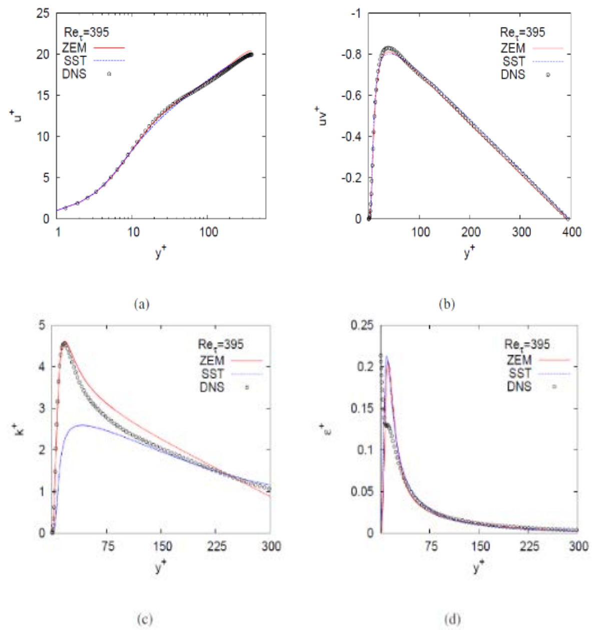

Figure 3: Developed channel flow at Re_T = 395: (a) mean velocity; (b) Reynolds shear stress; (c) kinetic energy of turbulence and (d) turbulent dissipation-rate

Simulations from the ZEM and Menter's SST model in wall units are presented in Figures 3 and 4. Comparisons are prepared by plotting results in terms of $u^+ = u/u_\tau$, $uv^+ = uv/u_\tau^2$, $k^+ = k/u_\tau^2$ and $\varepsilon^+ = \nu\varepsilon/u_\tau^4$ versus y^+. The Boussinesq approximation has been used to calculate the Reynolds shear stress. Figures 3(a) and 4(a) show velocity profiles from independent turbulence models. Simulations of the ZEM and SST model make good correspondence with DNS data. Noteworthily, acceptable congruence of the ZEM with DNS data can be observed without having transport and diffusion effects of the kinetic energy of turbulence and turbulent dissipation rate. Nonetheless, the SST model under-predicts the mean velocity profile in the wake-deficit region of boundary layer at Re_τ = 640. Perceptively, the SST k-ω model inherits this complication. Profiles of Reynolds shear stresses are plotted in Figures 3(b) and 4(b). As can be noticed, predictions of the ZEM and SST model agree well with DNS data; the difference between them is almost indistinguishable.

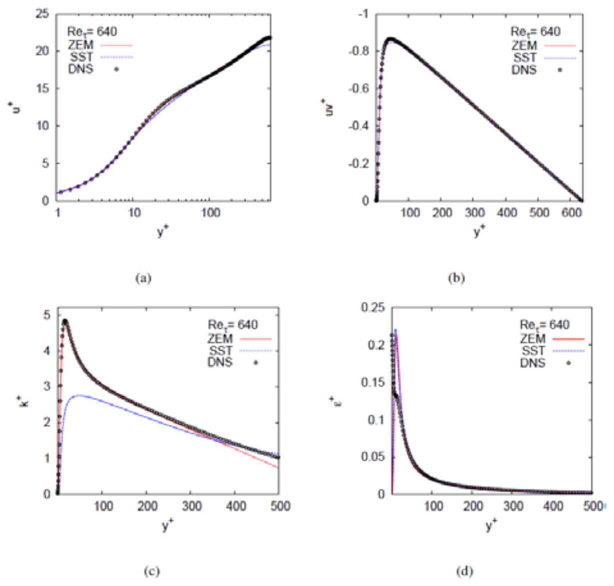

Figure 4: Developed channel flow at Re_T = 640: (a) mean velocity; (b) Reynolds shear stress; (c) kinetic energy of turbulence and (d) turbulent dissipation-rate

Model performances are assessed further with the k^+ profiles, shown in Figures 3(c) and 4(c). As can be observed, the ZEM matches with the DNS data in the near-wall region, whereas the k^+ profiles near the wall are badly under-predicted by the SST model. Figures 3(d) and 4(d) exhibit the ε^+ -profile from the ZEM and SST computations. It is well-known that experimental and DNS data exhibit a maximum ε^+ close to the wall. None of the turbulence models is capable of capturing a maximum ε^+ in the wall-vicinity; however, ε^+ profiles are predicted qualitatively well by both models after the wall regime. In fact, the numerical stability can be strengthened with such a behaviour of the ε^+ -profile in near-wall regions. The SST model experiences an enhancement in the convergence of numerical solvers due to this phenomenon.

4. CONCLUSIONS

Predicted results demonstrate that the ZEM has consistently outperformed the well-known SST model. The success of computation is facilitated by the essence of wall turbulence, comprising a universal multi-layer quantification of the stress length l_{12} which interprets the invariant (wall-normal distribution) of turbulent eddy-viscosity. The formulation of ZEM offers a positive perspective for the RANS model wherein an accurate prediction of wall-bounded flows is enhanced in collaboration with the fundamental understanding of wall turbulence. The ZEM presumably deserves the following specific merits: (a) may produce an accurate constraint on unsteady computations such as detached-eddy or large-eddy simulations; (b) may provide a calibration tool for experimental flow conditions and (c) flows over new geometries with available experimental data, the ZEM may be used to determine corresponding multi-layer parameters (like l_0^+ and y_{buf}^+) which specify the relevant flow physics of turbulent boundary-layer.

DISCLOSURE STATEMENT

No potential conflict of interest is reported by the authors.

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